

The Infrared Behaviour of the Pure Yang-Mills Green Functions

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Abstract

We study the infrared behaviour of the pure Yang-Mills correlators using relations that are well defined in the non-perturbative domain. These are the Slavnov-Taylor identity for three-gluon vertex and the Schwinger-Dyson equation for ghost propagator in the Landau gauge. We also use several inputs from lattice simulations. We show that lattice data are in serious conflict with a widely spread analytical relation between the gluon and ghost infrared critical exponents. We conjecture that this is explained by a singular behaviour of the ghost-ghost-gluon vertex function in the infrared. We show that, anyhow, this discrepancy is not due to some lattice artefact since lattice Green functions satisfy the ghost propagator Schwinger-Dyson equation. We also report on a puzzle concerning the infrared gluon propagator: lattice data seem to favor a constant non vanishing zero momentum gluon propagator, while the Slavnov-Taylor identity (complemented with some regularity hypothesis of scalar functions) implies that it should diverge.

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1 Introduction

1.1 Generalities

The whole set of correlation functions fully describes a Quantum Field Theory, as it is related to the S-matrix elements. In QCD or pure Yang-Mills theories Green functions are most often gauge dependent quantities which have no direct relationship with physical observables, the latter being necessarily gauge invariant. However, their indirect physical relevance is well known. In particular, long distance (or small momentum) Green functions will hopefully shed some light on the deepest mysteries of QCD such as confinement, spontaneous chiral symmetry breaking, etc. In this paper we concentrate our efforts on the study of the gluon and ghost Green functions at small momentum in a pure Yang-Mills theory. Our tools will be Slavnov-Taylor (ST) identities, Schwinger-Dyson (SD) equations and also several inputs from lattice QCD.

The infrared behaviour of Green functions has been extensively studied using different techniques, such as Schwinger-Dyson equations (see *e.g.* [3, 5, 24, 25] and references therein), renormalization group methods [26], stochastic quantization [4, 27]. These equations are exact consequences of QCD and can be easily derived using the path integral formalism. However their practical use reveals in most cases very difficult and one has to resort to a truncation which lessens the rigour of the method. One of the noticeable exceptions is the Schwinger-Dyson equation for the ghost propagator which contains only one integral and thus needs no truncation; this is the only one which we shall use in what follows. On the other hand, in order to exploit it in practice, one usually has to make appropriate *ansätze* for the gluon propagator and the ghost-ghost-gluon vertex.

We shall also use the Slavnov-Taylor identity which relates the three-gluons vertex to the ghost-ghost-gluon vertex in covariant gauges. Applying this relation in the non-perturbative domain will lead, under some assumptions about the infrared regularity of the dressing functions, to non trivial and surprising conclusions.

Lattice simulations are, of course, another major tool to study small momentum Green functions. However this paper is not meant to be a standard “lattice paper”. We aim at using SD and ST to derive properties of the small momentum Green functions and we will use lattice simulations as valuable inputs in our theoretical discussion and as a check of some hypotheses. As we shall see, the outcome proves to be quite surprising and undermines some widely spread beliefs.

In what follows we work in the Landau gauge, but some of the results we present are actually valid in any covariant gauge. Our notations are the following :

$$(F^{(2)})^{ab}(k) = -\delta^{ab} \frac{F(k^2)}{k^2} \quad (1)$$

$$(G_{\mu\nu}^{(2)})^{ab}(k) = \delta^{ab} \frac{G(k^2)}{k^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \quad (2)$$

$$\Gamma_{\mu\nu\rho}^{abc}(p, q, r) = f^{abc} \Gamma_{\mu\nu\rho}(p, q, r) \quad (3)$$

$$\begin{aligned}
\Gamma_\mu^{abc}(p, k; q) &= f^{abc}(-ip_\nu)g_0\tilde{\Gamma}_{\nu\mu}(p, k; q) \\
&= f^{abc}(-ip_\nu)g_0 \\
&\quad \cdot [\delta_{\nu\mu}a(p, k; q) - q_\nu k_\mu b(p, k; q) + p_\nu q_\mu c(p, k; q) + q_\nu p_\mu d(p, k; q) \\
&\quad + p_\nu p_\mu e(p, k; q)]
\end{aligned} \tag{4}$$

respectively for the ghost propagator, the gluon propagator, the three-gluons vertex and the ghost-ghost-gluon vertex³. All momenta are taken as entering. In eq. (4) $-p$ is the momentum of the outgoing ghost, k the momentum of the incoming one and $q = -p - k$ the momentum of the gluon. $F(p^2)$ and $G(p^2)$ are the dressing functions of the ghost and gluon propagators respectively. We parameterise the propagators in the infrared by setting at leading order

$$\begin{aligned}
G(p^2) &= \left(\frac{p^2}{\lambda^2}\right)^{\alpha_G} \\
F(p^2) &= \left(\frac{p^2}{\eta^2}\right)^{\alpha_F}, \quad \text{when } p^2 \text{ is small,}
\end{aligned} \tag{5}$$

where λ, η are some dimensional parameters.

Let us make a brief and partial summary of the present predictions for $\alpha_{F,G}$ ([3–5, 24–26, 28, 29]). We refer, for a very complete list of references, to the review by Alkofer and von Smekal [3]. All those references assume the relation $2\alpha_F + \alpha_G = 0$ and parameterise α_F and α_G as

$$\begin{aligned}
\alpha_F &= -\kappa_{\text{SD}} \\
\alpha_G &= 2\kappa_{\text{SD}}.
\end{aligned} \tag{6}$$

Different truncation schemes for the Schwinger-Dyson equations give ([3, 5, 24–26, 28, 29])

$$\begin{aligned}
\kappa_{\text{SD}} &= 0.92 & [29] \\
\kappa_{\text{SD}} &\in [0.17, 0.53] & [5], \text{ using a 2-loop perturbative input}
\end{aligned}$$

while another approach, ([4, 26]), predicts two possible solutions

$$\begin{aligned}
\kappa_{\text{SD}} &= 1, \quad \text{or} \\
\kappa_{\text{SD}} &= 0.59
\end{aligned}$$

Lattice simulations give $\alpha_G \simeq 1$. Note that $\alpha_G = 1$ corresponds to the gluon propagator being finite and non-zero at vanishing momentum; in other words, among the numbers we have just quoted, only the ones given in ref. [5] lead to a divergent infrared gluon propagator while the other values correspond to a vanishing one⁴.

1.2 Numerical setup of the lattice simulations

In this part we briefly describe the technical details of our numerical lattice simulations.

³We stick to the decomposition given in ref. [19] *except for the arguments of the scalar functions, for which we keep the same order as in Γ itself*

⁴After the completion of this paper our attention was drawn on ref. [6] which also predicts an IR-divergent gluon propagator and on refs. [7, 8] which lead to a finite non-zero one.

We use the standard Wilson action. For the $SU(2)$ gauge group we have used lattices of size 32^4 and 48^4 with $\beta = 2.3$. This value of β corresponds to $\beta_{SU(3)} \approx 5.75$. In the case of the $SU(3)$ gauge group the simulation has been done on 32^4 , 24^4 and smaller lattices with $\beta = 5.75$ and $\beta = 6.0$. Those rather low values of β have been chosen because they allow measurements at small momenta. We have used periodical boundary conditions for the gauge field. The gluon propagator is defined as a mean value over gauge field configurations :

$$< A_\mu^a(x) A_\nu^b(y) > .$$

The ghost propagator is calculated by the inversion of the discretised Faddeev-Popov operator (cf eq. 27). For this purpose we have used the conjugate gradient algorithm with the source

$$\left(1 - \frac{1}{V}, \frac{1}{V}, \dots, \frac{1}{V}\right)$$

where V is the number of lattice points. The rôle of the $1/V$ terms is to eliminate the zero modes of the Faddeev-Popov operator (corresponding to global gauge transformations) in order to allow its inversion in the orthogonal subspace (cf ref. [2]). All lattice data have been extrapolated to the continuum as described in ref. ([2]). A detailed report on all numerical results is presented in the same reference.

2 Constraints on α_F and α_G from the ghost Schwinger - Dyson equation

There is a widely used relation between α_F and α_G , referred to in the following as \mathbf{R}_α , which comes from the scaling analysis of the Schwinger-Dyson equation for the ghost propagator ([4]) and states that in four-dimensional space one has $2\alpha_F + \alpha_G = 0$.

We have attempted to test this relation on lattices with the characteristics indicated above. We plot in fig. 1 the quantity $F^2(p)G(p)$ as a function of p . If the relation \mathbf{R}_α was true, this quantity should be constant in the infrared domain. One can see that *it is not the case: F^2G goes to zero at small momenta*. On the other hand the ultraviolet (UV) behaviour is exactly the expected one. The same trend is already visible at β 's larger than ours : in refs. ([12, 13]) it is mentioned that F^2G might decrease as the momentum approaches zero⁵ at $\beta = 6.0$ and $\beta = 6.4$. The same authors have even reported on the same effect in the unquenched case [11]. Very recently Ilgenfritz et al. have published in ref. [16] results which go in the same direction (although the conclusions they draw thereof differ from ours).

It was suggested in [13] that Gribov copies might induce significant changes in the infrared behaviour of the ghost propagator. Could this explain our findings for F^2G ? In ref. [17] the accurate lattice gauge fixing (choosing the "best copy", corresponding to the lowest value of $\|A^{(g)}\|$) seems to lessen the infrared divergence of the ghost propagator [14, 15, 17], implying a further increase of the drop of F^2G in the infrared, while the gluon propagator is known to be only slightly affected by the presence of

⁵In ref. [12] it is assumed that the vertex function stays constant in the zero momentum limit, in which case F^2G is proportional to the strong coupling constant $\tilde{\alpha}_s$ in the MOM scheme based on ghost-ghost-gluon vertex.

lattice Gribov copies. Furthermore, we shall see that the SD equation is closely related to a lattice-SD equation which is a mathematical identity, valid independently of the choice of the Gribov copy. Thus the possible influence of Gribov copies on propagators cannot explain the behaviour of F^2G at small momenta.

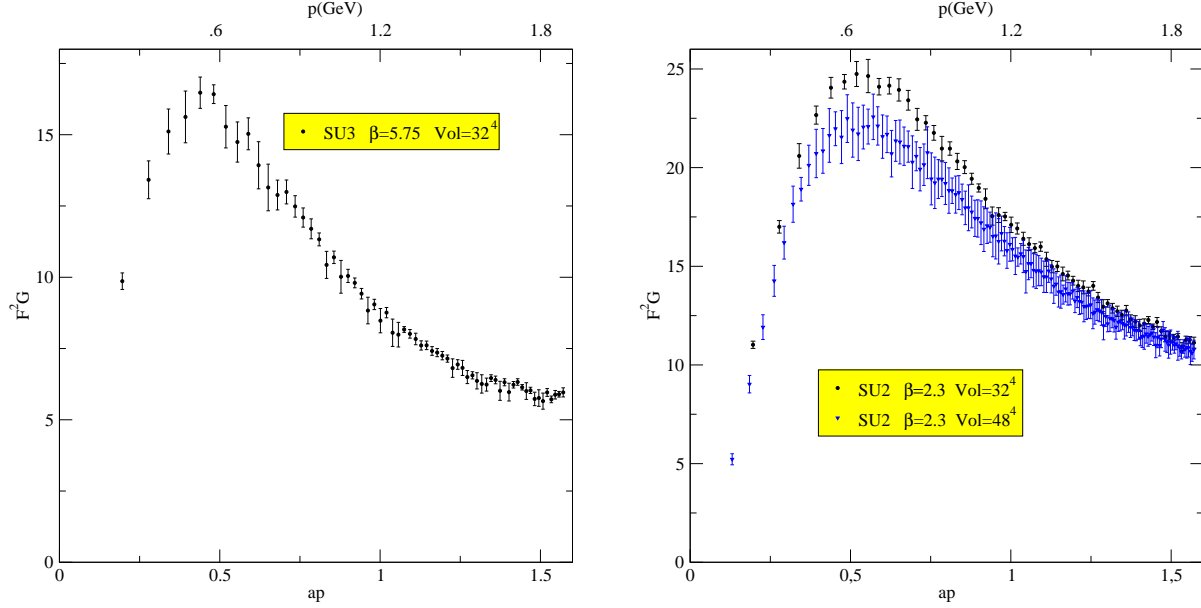


Figure 1: F^2G from lattice simulation for the $SU(3)$ (left, $32^4, \beta = 5.75$) and $SU(2)$ (right, 32^4 and $48^4, \beta_{SU(2)} = 2.3$) gauge groups. The β 's are chosen so as to give the same lattice spacings : 1.2 GeV^{-1} . If the relation $2\alpha_F + \alpha_G = 0$ was true this quantity should be constant in the infrared domain. One clearly does not see this behaviour.

The rest of this section is aimed at understanding this disagreement between lattice simulations and the theoretical claim (\mathbf{R}_α). We will first revisit the proof of the latter in order to identify all the hypotheses needed and submit each of them to a critical analysis. In the next-to-following section we will discuss a special writing of the ghost Schwinger-Dyson equation, in a form which involves only Green functions instead of vertices and can thus be directly tested on the lattice.

2.1 Revisiting the relation between α_F and α_G

We will examine to what extent the proof of \mathbf{R}_α is compelling, using the Schwinger-Dyson equation for the bare ghost propagator which can be written diagrammatically as

$$\left(\begin{array}{c} \text{---} \rightarrow \text{---} \\ a \quad k \quad b \end{array} \right)^{-1} = \left(\begin{array}{c} \text{---} \text{---} \rightarrow \text{---} \text{---} \\ a \quad k \quad b \end{array} \right)^{-1} - \begin{array}{c} \text{---} \text{---} \rightarrow \text{---} \text{---} \\ a, k \quad c, q \quad e \quad b, k \end{array}$$

i.e.

$$(F^{(2)})_{ab}^{-1}(k) = -\delta_{ab}k^2 \quad (7)$$

$$-g_0^2 f_{acd} f_{ebf} \int \frac{d^4 q}{(2\pi^4)} F_{ce}^{(2)}(q) (iq_{\nu'}) \Gamma_{\nu'\nu}(-q, k; q-k) (ik_{\mu}) (G^{(2)})_{\mu\nu}^{fd}(q-k),$$

where we use the notations of eqs. (1-5) and g_0 is the bare coupling constant. This integral equation is written in terms of bare Green functions. It can be cast into a renormalized form by multiplying $G^{(2)}$ (resp. $F^{(2)}, \Gamma$) by Z_3^{-1} (resp. $\tilde{Z}_3^{-1}, \tilde{Z}_1$) and g_0^2 by $Z_g^{-2} = Z_3 \frac{\tilde{Z}_3^2}{\tilde{Z}_1^2}$ and multiplying the k^2 term by \tilde{Z}_3 . The integral therein is ultraviolet divergent but one can check that the cut-off dependence is matched by the cut-off dependence of Z_g and of the \tilde{Z}_3 factor multiplying k^2 . Later on we will only use *subtracted* SD equations such that the UV divergence is cancelled as well as the $\tilde{Z}_3 k^2$ term. These subtracted SD equations hold both in terms of bare and renormalized Green functions without any explicit renormalization factor.

Let us now consider (7) at small momenta k . The ghost-gluon vertex may be expressed as

$$q_{\nu'} \tilde{\Gamma}_{\nu'\nu}(-q, k; q-k) = q_{\nu} H_1(q, k) + (q-k)_{\nu} H_2(q, k) \quad (8)$$

where, using the decomposition (4), one gets:

$$\begin{aligned} H_1(q, k) &= a(-q, k; q-k) - (q^2 - q \cdot k) (b(-q, k; q-k) \\ &\quad + d(-q, k; q-k)) + q^2 e(-q, k; q-k) \simeq \\ &\stackrel{k \rightarrow 0}{\simeq} a(-q, k; q-k) - q^2 (b(-q, k; q-k) + d(-q, k; q-k) - e(-q, k; q-k)) \\ H_2(q, k) &= (q^2 - q \cdot k) b(-q, k; q-k) - q^2 c(-q, k; q-k) \end{aligned} \quad (9)$$

In the Landau gauge, because of the transversality condition, H_2 does not contribute. Thus, dividing both sides of eq. (7) by k^2 and omitting colour indices, one obtains

$$\frac{1}{F(k)} = 1 + g_0^2 N_c \int \frac{d^4 q}{(2\pi)^4} \left(\frac{F(q^2) G((q-k)^2)}{q^2 (q-k)^2} \left[\frac{\frac{(k \cdot q)^2}{k^2} - q^2}{(q-k)^2} \right] H_1(q, k) \right). \quad (10)$$

2.2 What does the “non-renormalization theorem” exactly say?

A widely used statement, known as the “non-renormalization theorem”, claims that, in the Landau gauge, the renormalization constant \tilde{Z}_1 of the ghost gluon vertex is exactly one. Note that there is no reference to a particular renormalisation scheme. Formulated in this way, this claim is wrong. Let us first state and then explain below what is true in our opinion :

1) There is a true and very clear statement which can be extracted from Taylor’s paper (the argument is given below), ref. [1].

$$\widetilde{\Gamma_{\mu}^{abc, Bare}}(-p, 0; p) = -i f^{abc} p_{\mu} \quad (11)$$

i.e. there is no radiative correction in this particular momentum configuration (with zero momentum of the ingoing ghost)

2) This entails that $\tilde{\Gamma}_\mu^{abc,Bare}(p, k; q)$ is *finite* whatever the external momenta, and that therefore $\tilde{Z}_1^{\overline{\text{MS}}} = 1$. In addition, we get also trivially $\tilde{Z}_1^{\text{MOM}_h} = 1$, where MOM_h refers to the configuration of momenta in equation (11). In general, in other schemes, *there is* a finite renormalisation, and this is why we do not adopt the misleading expression "non-renormalization theorem".

3) In particular, one finds in the very extensive calculations of radiative corrections at least two cases of MOM schemes where there is a finite renormalisation (and certainly many more) : MOM_g in the notations of ref. [19], and the *symmetric* MOM scheme. For the latter, we give the proof below.

The essence of Taylor's argument is actually very simple. In a kinematical situation where the incoming ghost momentum is zero, consider any perturbative contribution to the ghost-gluon vertex. Following the ghost line in the direction of the flow, the first vertex will be proportional to the outgoing ghost momentum p_μ , i.e. to the gluon momentum $-p_\mu$. In the Landau gauge this contribution will thus give 0 upon contraction with the gluon propagator $D_{\mu\nu}(p)$. Therefore the only contribution to remain is the tree-level one. In other words the bare ghost-gluon vertex is shown to be equal to its tree-level value in these kinematics : $\tilde{\Gamma}_\mu^{abc,Bare}(-p, 0; p) = -if^{abc}p_\mu$. This result has been checked by means of a direct evaluation to three loops in perturbation theory by Chetyrkin. In our notations :

$$H_1(p, 0) + H_2(p, 0) = 1.$$

Note that in the Schwinger-Dyson equation 10, only H_1 is present, and the theorem of Taylor does not tell that $H_1(p, 0) = 1$, as seems assumed in many Schwinger-Dyson calculations, where it plays a crucial rôle in the proof of \mathbf{R}_α .

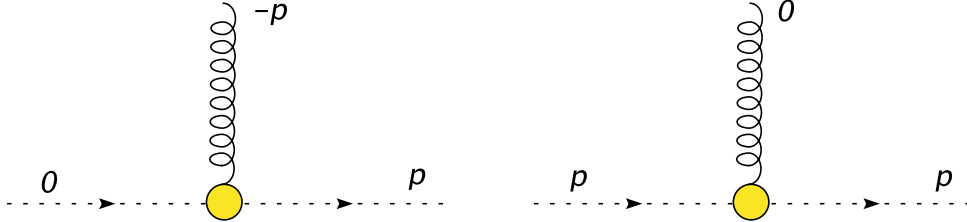


Figure 2: The kinematical situations considered below. The left diagram (0-momentum incoming ghost) corresponds to Γ_h below which is known to be equal to one. The right one (0-momentum gluon) corresponds to Γ_g and leads to a non-trivial p^2 -dependence

As an illustration of our point 3), let us quote the formulas from the appendix of ref. [19], reduced to the situation we are interested in ($\xi_L = 0, n_f = 0$). The two dressing functions $\tilde{\Gamma}_h$ (resp. $\tilde{\Gamma}_g$) are defined by $\tilde{\Gamma}_\mu^{abc}(-p, 0, p) = -if^{abc}\tilde{\Gamma}_h(p)$ (resp. $\tilde{\Gamma}_\mu^{abc}(-p, p, 0) = -if^{abc}\tilde{\Gamma}_g(p)$) and correspond to the kinematical situations depicted in the left (resp. right) part of fig. (2). We have already mentionned that Γ_h is exactly

one, but this does not hold for Γ_g and, indeed, one has at three loops :

$$\begin{aligned}\tilde{\Gamma}_g^{\text{MS}}|_{p^2=\mu^2} &= 1 + \frac{3}{4} \frac{\alpha_s}{4\pi} C_A + \frac{599}{96} \left(\frac{\alpha_s}{4\pi}\right)^2 C_A^2 + \left[\frac{43273}{432} + \frac{783}{64} \zeta_3 - \frac{875}{64} \zeta_5 \right] \left(\frac{\alpha_s}{4\pi}\right)^3 C_A^3 \\ &+ \left[\frac{27}{4} - \frac{639}{16} \zeta_3 + \frac{225}{8} \zeta_5 \right] \left(\frac{\alpha_s}{4\pi}\right)^3 C_A^2 C_F .\end{aligned}\quad (12)$$

It is then easy to find the p^2 -dependence :

$$\tilde{\Gamma}_g = \tilde{\Gamma}_g^{\text{MS}}|_{p^2=\mu^2} + \left[\frac{11}{4} C_A^2 \left(\frac{\alpha_s}{4\pi}\right)^2 + \frac{7813}{144} C_A^3 \left(\frac{\alpha_s}{4\pi}\right)^3 + \dots \right] \log\left(\frac{\mu^2}{-p^2}\right) + \dots$$

In ref. [28] the non -renormalization theorem is understood as the statement that the vertex reduces to its tree-level form at all symmetric-momenta points in a symmetric subtraction scheme. However this statement is not supported by a direct evaluation. Using the one-loop results of Davydychev (ref. [20]) one gets in a symmetric configuration the value

$$\tilde{\Gamma}_\mu^{abc}(p, k; q)|_{p^2=k^2=q^2=\mu^2} = -i f^{abc} \left\{ p_\mu \left(1 + \frac{\alpha_s}{4\pi} \frac{C_A}{12} (9 + \frac{5}{2}\phi) \right) + q_\mu \frac{\alpha_s}{4\pi} \frac{C_A}{12} (3 + \frac{5}{4}\phi) \right\} \quad (13)$$

with $\phi = \frac{4}{\sqrt{3}} Cl_2(\frac{\pi}{3})$, $Cl_2(\frac{\pi}{3}) = 1.049\dots$.

According to ref. [28] the coefficient of p_μ should be one. The presence of α_s in the above formulas implies on the contrary that the vertex will in general depend on the momenta : using the results given in the appendices of ref. [19] one finds for the leading p^2 -dependence $-i f^{abc} \left\{ \frac{11}{3} \frac{C_A^2}{12} \left(\frac{\alpha_s}{4\pi}\right)^2 \log\left(\frac{p^2}{\mu^2}\right) ((9 + \frac{5}{2}\phi)p_\mu + (3 + \frac{5}{4}\phi)q_\mu) \right\}$. This dependence is logarithmic, as is expected in a perturbative approach. Furthermore, in ref. ([28]) it is supposed that the vertex function takes the form $(q^2)^\ell (k^2)^m ((q-k)^2)^n$ with the restriction $\ell + m + n = 0$. One should note that this last condition corresponds in our notations to $\alpha_\Gamma = 0$ (cf. section (2.3) below). This restriction comes from the assumption that the symmetric vertex is equal to 1 for any p^2 , which, as we have just seen, is actually not the case. Therefore we shall adopt a more general point of view and keep open the possibility of a non perturbative effect leading to a singular or vanishing limit of H_1 when $q \rightarrow 0$. We should mention that the problem of the p^2 -dependence of the ghost-gluon vertex has already been addressed in refs. [33, 34]. However these authors work under the condition \mathbf{R}_α which appears not to be satisfied by our lattice data.

2.3 A subtracted Schwinger-Dyson equation

Let us now consider two infrared scales $k_1 \equiv k$ and $k_2 \equiv \kappa k$. Calculating the difference of eq. (10) taken at scales k_1 and k_2 and supposing for the moment that $\alpha_F \neq 0$ one obtains

$$\begin{aligned}\frac{1}{F(k)} - \frac{1}{F(\kappa k)} &\propto (1 - \kappa^{-2\alpha_F})(k^2)^{-\alpha_F} = g_0^2 N_c \int \frac{d^4 q}{(2\pi)^4} \left(\frac{F(q^2)}{q^2} \left(\frac{(k \cdot q)^2}{k^2} - q^2 \right) \right. \\ &\times \left. \left[\frac{G((q-k)^2) H_1(q, k)}{((q-k)^2)^2} - \frac{G((q-\kappa k)^2) H_1(q, \kappa k)}{((q-\kappa k)^2)^2} \right] \right).\end{aligned}\quad (14)$$

We now make the hypothesis that there exists a scale q_0 such that

$$G(q^2) \sim (q^2)^{\alpha_G}, \quad F(q^2) \sim (q^2)^{\alpha_F}, \quad \text{for } q^2 \leq q_0^2.$$

Similarly, we suppose that H_1 can be written for $q^2, k^2 \leq q_0^2$ as

$$H_1(q, k) \sim (q^2)^{\alpha_\Gamma} h_1\left(\frac{q \cdot k}{q^2}, \frac{k^2}{q^2}\right)$$

or as

$$H_1(q, k) \sim ((q - k)^2)^{\alpha_\Gamma} h_2\left(\frac{q \cdot k}{q^2}, \frac{k^2}{q^2}\right)$$

where the scalar functions $h_{1,2}$ are supposed to be regular enough (i.e. free of singularities worse than logarithmic) and expandable in Taylor series for $k \rightarrow 0$. They are obviously invariant under any simultaneous rescaling of both q and k . The exponent α_Γ gives the leading critical behaviour of H_1 on q .

Thus we rewrite (14) by rescaling $k \rightarrow \lambda k$ with λ chosen so that $(\lambda k)^2 \ll q_0^2$ and splitting the integral in the r.h.s. into two parts

$$I_1(\lambda) = \int_{q^2 < q_0^2} \frac{d^4 q}{(2\pi)^4} [\dots], \quad I_2(\lambda) = \int_{q^2 > q_0^2} \frac{d^4 q}{(2\pi)^4} [\dots]. \quad (15)$$

In I_1 , since $(\lambda k)^2 \ll q_0^2$, we can substitute the infrared approximations (5) for G and F . I_1 is infrared convergent if :

$$\begin{aligned} \alpha_F + \alpha_\Gamma &> -2 && \text{IR convergence at } q^2 = 0 \\ \alpha_G + \alpha_\Gamma &> -1 && \text{IR convergence at } (q - k)^2 = 0 \text{ and } (q - \kappa k)^2 = 0 \end{aligned} \quad (16)$$

We shall suppose in the following that these conditions are verified. We then obtain, performing the change of variable $q \rightarrow \lambda q$ and writing generically h for $h_{1,2}$:

$$\begin{aligned} I_1(\lambda) &\approx \lambda^{2(\alpha_F + \alpha_G + \alpha_\Gamma)} \int_{q^2 < \frac{q_0^2}{\lambda^2}} \frac{d^4 q}{(2\pi)^4} \left((q^2)^{\alpha_F + \alpha_\Gamma - 1} \left(\frac{(k \cdot q)^2}{k^2} - q^2 \right) \right. \\ &\times \left. \left[((q - k)^2)^{\alpha_G - 2} h\left(\frac{q \cdot k}{q^2}, \frac{k^2}{q^2}\right) - ((q - \kappa k)^2)^{\alpha_G - 2} h\left(\kappa \frac{q \cdot k}{q^2}, \kappa^2 \frac{k^2}{q^2}\right) \right] \right). \end{aligned} \quad (17)$$

The point we have to keep in mind is the fact that the upper bound of the integral goes to infinity when $\lambda \rightarrow 0$. This potentially induces a dependence on λ whose interplay with the behaviour explicitly shown in (17) we must check. In this limit, the convergence of the integral depends on the asymptotic behaviour of the whole integrand for large q . In particular, the leading contribution of the square bracket in eq. (17) behaves as

$$(q^2)^{\alpha_G - 2} \left(\frac{k^2}{q^2} \right) \sim q^{2\alpha_G - 6},$$

because the terms in $q \cdot k$, being odd under $q_\mu \rightarrow -q_\mu$, give a null contribution under the angular integration in eq. (17). Thus, assuming the conditions (16) are satisfied

Case	$\alpha_F \neq 0$ $\alpha_F + \alpha_G + \alpha_\Gamma < 1$	$\alpha_F \neq 0$ $\alpha_F + \alpha_G + \alpha_\Gamma \geq 1$	$\alpha_F = 0$ $\alpha_G + \alpha_\Gamma < 1$	$\alpha_F = 0$ $\alpha_G + \alpha_\Gamma \geq 1$
IR Convergence Conditions	$\alpha_F + \alpha_\Gamma > -2$ $\alpha_G + \alpha_\Gamma > -1$	$\alpha_F + \alpha_\Gamma > -2$ $\alpha_G + \alpha_\Gamma > -1$	$\alpha_\Gamma > -2$ $\alpha_G + \alpha_\Gamma > -1$	$\alpha_\Gamma > -2$ $\alpha_G + \alpha_\Gamma > -1$
SD constraints	$2\alpha_F + \alpha_G + \alpha_\Gamma = 0$	$\alpha_F = -1$	excluded	none

Table 1: Summary of the various cases regarding the α coefficients

the integral $I_1(\lambda)$ is guaranteed to be convergent when $q \rightarrow 0$ (or $k \rightarrow q$) and its asymptotics for small λ is given by

$$I_1(\lambda) \sim \begin{cases} \lambda^{2(\alpha_G + \alpha_F + \alpha_\Gamma)} \int_0^{q_0/\lambda} dq q^{2(\alpha_F + \alpha_G + \alpha_\Gamma) - 3} \sim \lambda^{2(\alpha_G + \alpha_F + \alpha_\Gamma)} & \text{if } \alpha_G + \alpha_F + \alpha_\Gamma < 1 \\ \lambda^2 \int_0^{q_0} dq q^{2(\alpha_F + \alpha_G + \alpha_\Gamma) - 3} \sim \lambda^2 & \text{if } \alpha_G + \alpha_F + \alpha_\Gamma \geq 1 \end{cases} \quad (18)$$

because in both cases the integral on the momentum q is finite and does not depend on λ in the limit $\lambda \rightarrow 0$.

Let us now consider I_2 . Its dependence on λ is explicit in the factor

$$\frac{G((q - \lambda k)^2) H(q, \lambda k)}{((q - \lambda k)^2)^2} - \frac{G((q - \lambda \kappa k)^2) H(q, \lambda \kappa k)}{((q - \lambda \kappa k)^2)^2}$$

which stems from the substitution $k \rightarrow \lambda k$ in (14). Clearly, this quantity can only be even in λ : any odd power of λ would imply an odd power of $q \cdot k$ whose angular integral is zero. Since the integrand is identically zero at $\lambda = 0$ and the integral is ultraviolet convergent, it is proportional to λ^2 (unless some accidental cancellation forces it to behave as a higher even power of λ).

So, if the first of the conditions (18) is verified, it follows $I_1 + I_2 \approx \lambda^{2(\alpha_F + \alpha_G + \alpha_\Gamma)}$, else $I_1 + I_2 \approx \lambda^2$. Comparing this to the left hand side of eq. (10)

$$\frac{1}{F(\lambda k)} - \frac{1}{F(\lambda \kappa k)} \sim \lambda^{-2(\alpha_F)} \quad (19)$$

we then conclude :

$$\begin{aligned} \alpha_F + \alpha_G + \alpha_\Gamma < 1 &\implies 2\alpha_F + \alpha_G + \alpha_\Gamma = 0 \\ \alpha_F + \alpha_G + \alpha_\Gamma \geq 1 &\implies \alpha_F = -1 \end{aligned} \quad (20)$$

In the particular case $\alpha_F = 0$ the leading term in the l.h.s. of eq. (14) is identically zero. We are left with the subleading one which, pursuing the same argumentation, we suppose to be proportional to k^2 . Then the argument is the same as in the previous case except for the power of λ in the r.h.s. of eq. (19) which becomes equal to 2. It results that the case $\alpha_G + \alpha_\Gamma < 1$ is now excluded while the case $\alpha_G + \alpha_\Gamma \geq 1$ provides no extra constraint.

The various possibilities which have appeared in this discussion are summarised in table 1. From this table it appears that *only the triple condition that*

$$\alpha_F \neq 0, \quad \alpha_\Gamma = 0, \quad \alpha_F + \alpha_G < 1$$

does actually imply the *standard statement that* $2\alpha_F + \alpha_G = 0$. However, the plot in Fig.1 indicates a behaviour $2\alpha_F + \alpha_G > 0$, indicating that at least one of these conditions is not fulfilled.

Let us assume for the moment that $\alpha_F + \alpha_G \geq 1$ and $\alpha_\Gamma = 0$. Then $2\alpha_F + \alpha_G \geq 0$, in agreement with fig.1, and $\alpha_G \geq 2$. However the possibility that α_G be greater than 2 is unambiguously excluded by the lattice simulations so that the hypothesis has to be rejected. Furthermore, as we shall see, one can derive from the Slavnov-Taylor identity relating the three-gluons and ghost-ghost-gluon vertices the inequality $\alpha_G < 1$ if one assumes that some of the scalar form factors of these vertices are regular when one momentum goes to zero.

We now consider the hypothesis $2\alpha_F + \alpha_G + \alpha_\Gamma = 0$ with $\alpha_\Gamma < 0$ to comply with the lattice indications of fig.1. This implies that some of the scalar factors of the ghost-ghost-gluon vertex are singular in the infrared. We shall turn back to this possibility in the concluding remarks of this section (2.4.4). The question is whether a non-perturbative effect could generate a non vanishing α_Γ .

A direct lattice estimate of the ghost-ghost-gluon vertex would be welcome. However this is a difficult task. A direct measurement implying a zero momentum ghost is impossible since the corresponding Green functions are singular because of the zero modes of the Faddeev-Popov operator. A careful limiting procedure implying very small external momenta has to be performed. This study is under way. In between we propose a simpler check based on another writing of the SD equation (7) in terms of a pinched Green function which can be directly checked on the lattice. This will also allow to control the lattice artefacts and to rule out the hypothesis that the problem encountered in fig. 1 is simply due to a lattice artefact.

2.4 Green function formulation of the ghost SD equation

We will now rewrite the ghost SD equation using only propagators and the ghost-ghost-gluon Green function. In this form, its validity can be tested on the lattice, because what one directly calculates in lattice simulations are *Green functions*, not *vertex functions*. If we consider the loop integral in Eq. (7) we can see that it is nothing else but a ghost-ghost-gluon Green function in which the left ghost leg has been cut and where the gluon and right-hand ghost have been pinched onto the same point in configuration space. We shall see that this quantity is directly accessible from lattice data *without making any specific assumption about the behaviour of the vertex function*.

The interest of this approach is that it will help us to throw a closer look at the compatibility of the lattice simulations and the SD equations. Indeed, as we have just seen, we are facing a contradiction between, on the one hand the lattice estimate of $F^2 G$ (fig. 1) and, on the other hand, the relation \mathbf{R}_α which is derived from the SD equation (7) complemented by a regularity assumption ($\alpha_\Gamma = 0$) suggested by perturbation theory. Therefore we feel the need to directly confront lattice calculations with SD. The form of SD which is presented in this subsection allows such a direct confrontation and, this form being closely related to a lattice SD equation which is just a mathematical identity, we are in a good position to trace back any discrepancy.

In the next subsection we present the continuum limit derivation of the Green

function formulation of the ghost SD, as well as its lattice derivation.

2.4.1 Continuum limit case

For a given gauge field configuration $\mathcal{A}_\mu = A_\mu^a t^a$, the Faddeev-Popov operator in covariant gauges is given by

$$M(x, y)_{\text{1conf}} = (\partial_\mu D_\mu) \delta(x - y) = (\Delta + ig_0 \partial_\mu \mathcal{A}_\mu) \delta^4(x - y) \equiv F_{\text{1conf}}^{(2)-1}(\mathcal{A}, x, y), \quad (21)$$

and it is equal to the inverse of the ghost correlator in the background of the gluon field \mathcal{A} , $F_{\text{1conf}}^{(2)}$. The subscript means here that the equation is valid for any given gauge configuration. This can be written

$$\delta^4(x - y) \equiv M_{\text{1conf}}(x, z) F_{\text{1conf}}^{(2)}(\mathcal{A}, z, y), \quad (22)$$

where a summation on z is understood. Expanding M according to (21) we get

$$\delta(x - y) = \Delta(x, z) F_{\text{1conf}}^{(2)}(z, y) + ig_0 \partial_\mu^{(x)} \left(\mathcal{A}_\mu(x) F_{\text{1conf}}^{(2)}(\mathcal{A}, x, y) \right), \quad (23)$$

valid for *any* gauge field configuration. Performing the path integral one gets the mean value on gauge configurations

$$\delta(x - y) = \Delta(x, z) \langle F_{\text{1conf}}^{(2)}(z, y) \rangle + ig_0 \partial_\mu^{(x)} \langle \mathcal{A}_\mu(x) F_{\text{1conf}}^{(2)}(\mathcal{A}, x, y) \rangle. \quad (24)$$

Of course $\langle F_{\text{1conf}}^{(2)}(z, y) \rangle$ is nothing else but the ghost propagator defined in equation (1).

The averages $\langle F^{(2)}(x, y) \rangle$ and $\partial_\mu^{(x)} \langle \mathcal{A}_\mu(x) F^{(2)}(x, y) \rangle$ are invariant under translations so that one can replace the derivative $\partial_\mu^{(x)} \rightarrow -\partial_\mu^{(y)}$. We take $x = 0$ and perform the Fourier transform on the y variable (note that there is no tilde on $\mathcal{A}_\mu(0)$)

$$1 = -p^2 \tilde{F}^2(p) - g_0 p_\mu \langle \mathcal{A}_\mu(0) \tilde{F}_{\text{1conf}}^{(2)}(\mathcal{A}, p) \rangle. \quad (25)$$

Finally we get

$$F(p^2) = 1 + g_0 \frac{p_\mu}{N_c^2 - 1} f^{abc} \langle A_\mu^c(0) \tilde{F}_{\text{1conf}}^{(2)ba}(\mathcal{A}, p) \rangle. \quad (26)$$

One caveat is in order here. Eq. (7) implies an ultraviolet divergent integral which is matched by renormalization constants. This divergence is of course also present in eq. (24) via the local product of operators at x . In section 2.1 this divergence was canceled by subtracting two terms as is apparent in eq. (14). In the following this divergence is regularised by the lattice cut-off. To perform the connection with the discussion in section 2.1 it will be necessary to perform an analogous subtraction when using the form (26).

2.4.2 Lattice case

We now repeat the same steps as in the preceding paragraph for the lattice version of the Faddeev-Popov operator

$$\begin{aligned} M_{xy}^{ab} \equiv F_{\text{1conf}}^{(2)}(U, x, y)^{-1} &= \sum_\mu \left[S_\mu^{ab}(x) (\delta_{x,y} - \delta_{y,x+e_\mu}) - S_\mu^{ab}(x - e_\mu) (\delta_{y,x-e_\mu} - \delta_{y,x}) + \right. \\ &\quad \left. + \frac{1}{2} f^{abc} [A_\mu^c(x) \delta_{y,x+e_\mu} - A_\mu^c(x - e_\mu) \delta_{y,x-e_\mu}] \right], \end{aligned} \quad (27)$$

where

$$\begin{aligned} S_\mu^{ab}(x) &= -\frac{1}{2}\text{Tr} \left[\{t^a, t^b\} (U_\mu(x) + U_\mu^\dagger(x)) \right] \\ A_\mu(x) &= \frac{U_\mu(x) - U_\mu^\dagger(x)}{2} - \frac{1}{N}\text{Tr} \frac{U_\mu(x) - U_\mu^\dagger(x)}{2}, \end{aligned} \quad (28)$$

in which $U_\mu(x)$ denotes a standard link variable⁶, and e_μ is a unitary vector in direction μ . We define

$$\Delta_U = \sum_\mu \left(S_\mu^{ab}(x) (\delta_{x,y} - \delta_{y,x+e_\mu}) - S_\mu^{ab}(x - e_\mu) (\delta_{y,x-e_\mu} - \delta_{y,x}) \right). \quad (29)$$

The appearance of Δ_U as the appropriate discretisation of the usual Laplacian operator Δ is dictated by the gauge invariance of the original Yang-Mills action, which imposes that the standard ∇ operator be replaced by its *covariant* version and by the specific form $-\Re\text{Tr}(\sum_{\text{links}} U^g)$ – of the functional to be minimized in order to fix the Landau gauge.

Then, multiplying M_{xy}^{ab} by $F^{(2)}(x, y)$ from the right, one obtains :

$$\begin{aligned} \frac{1}{N_c^2 - 1} \text{Tr} \Delta_U(y, z) F_{1\text{conf}}^{(2)}(U; z, u) &= \delta_{y,u} - \\ &- \frac{f^{abc}}{2(N_c^2 - 1)} \left[A_\mu^c(y) F_{1\text{conf}}^{(2)ba}(U; y + e_\mu, u) - A_\mu^c(y - e_\mu) F_{1\text{conf}}^{(2)ba}(U; y - e_\mu, u) \right], \end{aligned} \quad (30)$$

This is an exact mathematical identity for each gauge configuration which must actually be fulfilled by our lattice data since our $F_{1\text{conf}}^{(2)}$ are computed by means of an explicit inversion of the Faddeev-Popov operator. From this fact results an important feature which we wish to stress : *since eq. 30 is valid in any configuration, its consequences are free of any ambiguity originating from the presence of Gribov copies.* Upon averaging over the configurations one gets

$$\begin{aligned} \frac{1}{N_c^2 - 1} \text{Tr} \langle \Delta_U(y, z) F_{1\text{conf}}^{(2)}(z, u) \rangle &= \delta_{y,u} - \\ &- \frac{f^{abc}}{2(N_c^2 - 1)} \left[\langle A_\mu^c(y) F_{1\text{conf}}^{(2)ba}(U, y + e_\mu, u) - A_\mu^c(y - e_\mu) F_{1\text{conf}}^{(2)ba}(U, y - e_\mu, u) \rangle \right] \end{aligned} \quad (31)$$

Of course, this averaging procedure depends on the way chosen to treat Gribov's problem : the particular set of configurations over which it is performed depends on the prescription which is adopted (choice of any local minimum of A^2 , restriction to the fundamental modular region...) and, consequently, the Green functions will vary but *they will in any case satisfy the above equation.* Like in the continuum case, after setting y to zero, a Fourier transformation with respect to u gives:

$$\begin{aligned} \frac{1}{N_c^2 - 1} \text{Tr} \sum_u e^{ip \cdot u} \langle \Delta_U(0, z) F_{1\text{conf}}^{(2)}(U, z, u) \rangle &= 1 - \\ &- i \sin(p_\mu) \frac{f^{abc}}{(N_c^2 - 1)} \langle A_\mu^c(0) \tilde{F}_{1\text{conf}}^{(2)ba}(U, p) \rangle \end{aligned} \quad (32)$$

Note that although eqs. (30) and (31) have to be exactly verified by lattice data eq. (32) does only approximately (within statistical errors) since it relies on translational invariance, which could be guaranteed only if we used an infinite number of configurations.

⁶Note that the definition of A_μ given in (28) differs from the naïve one by a factor ig_0 . This is the reason for the presence of i and the absence of g_0 in eq. (31) as compared to eq. (10).

Eq. (32) is a discretised version of (26). Therefore any lattice correlator, satisfying (32), should also satisfy (26) *up to non-zero lattice spacing effects*. Among the various sources for such effects the use of the specific Δ_U discretisation of the Laplacian operator in the l.h.s. deserves some comments. The gauge fields present in the $S_\mu^{ab}(x)$ terms in eq. (29) generate in Δ_U the so-called “tadpole” diagrams such as in fig. 3. According

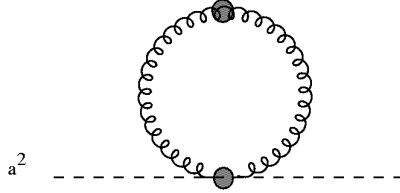


Figure 3: Example of the terms in the Schwinger-Dyson equation on the lattice.

to the philosophy developed by Lepage and Mackenzie in ref. ([23]) the tadpole contribution can be estimated by a mean field method. Using the average plaquette P (for $\beta = 6.0$ $P \simeq 0.5937$) one predicts a tadpole correction factor $\propto P^{-(1/4)} \simeq 1.14$. These terms disappear in the continuum limit but they do so only very slowly : the tadpole corrections (1 - plaquette) vanish only as an inverse logarithm with the lattice spacing. This is to be contrasted with the corrections arising in the r.h.s which are expected to be of order a^2 .

2.4.3 Checking the validity of the SD equation on the lattice

Since, as we have just mentioned, eqs. (30) and (31) are mathematical identities, there is in principle little –if any– to learn from a verification of eq. (32), except for the verification that our configurations are actually in the Landau gauge. On the other hand one thing we wish to be reassured about is the possible role of lattice artefacts in the discrepancies we have noticed.

We begin with a comparison of the continuum r.h.s of eq. (26) with the l.h.s. of (32). Both sides are plotted in fig. (4).

The agreement is impressive. Should we on the contrary use eq. (26) itself we observe a clear disagreement between the two sides of the equation. What we thus learn is that the major part of the discretization artefact comes from Δ_U . In fact, our lattice data show that $\Delta_U \simeq \Delta/1.16$ *almost independently of the momentum*, see fig. 5. This is in good agreement with the correction factor of 1.14 obtained from Lepage-Mackenzie’s mechanism [23]. To conclude the tadpole effect explains almost all of the discrepancy observed when trying to verify (26). One can also understand why this discretisation artefact is so large : this is due to the slow logarithmic vanishing of the tadpole corrections.

It results from this discussion that the lattice artefacts cannot be blamed for the violation of the relation \mathbf{R}_α observed in fig. (1) : the tadpole effect has been seen to produce a corrective factor almost constant in p , thus unable to explain an error in the power behaviour.

2.4.4 Concluding remarks

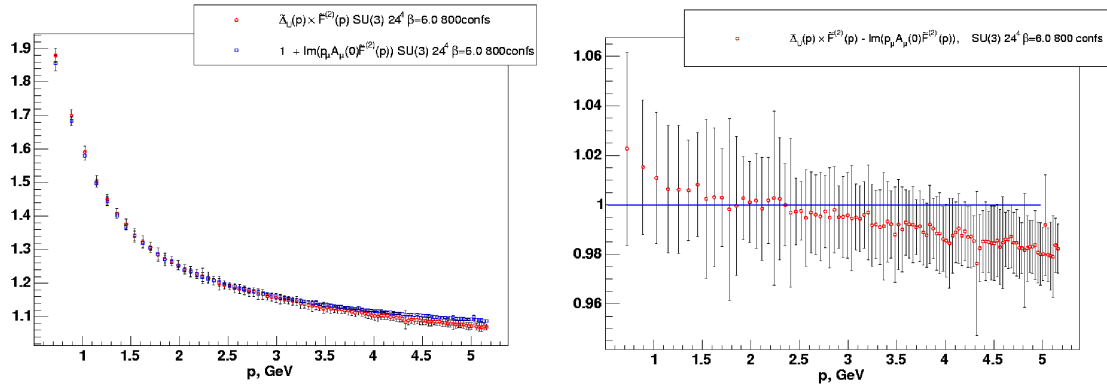


Figure 4: Check of the validity of the SD equation on the lattice. Left : $\frac{1}{N_c^2-1} \langle \Delta_U(p) \text{Tr} \tilde{F}_{1\text{conf}}^{(2)}(p) \rangle$ (circles) is plotted vs. $1 + g_0 \frac{p_\mu}{N_c^2-1} f^{abc} \langle A_\mu^c(0) \tilde{F}_{1\text{conf}}^{(2)ba}(\mathcal{A}, p) \rangle$ (squares). Right : $\frac{1}{N_c^2-1} \langle \Delta_U(p) \text{Tr} \tilde{F}^{(2)}(p) \rangle - g_0 \frac{p_\mu}{N_c^2-1} f^{abc} \langle A_\mu^c(0) \tilde{F}_{1\text{conf}}^{(2)ba}(\mathcal{A}, p) \rangle$ is compared to 1.

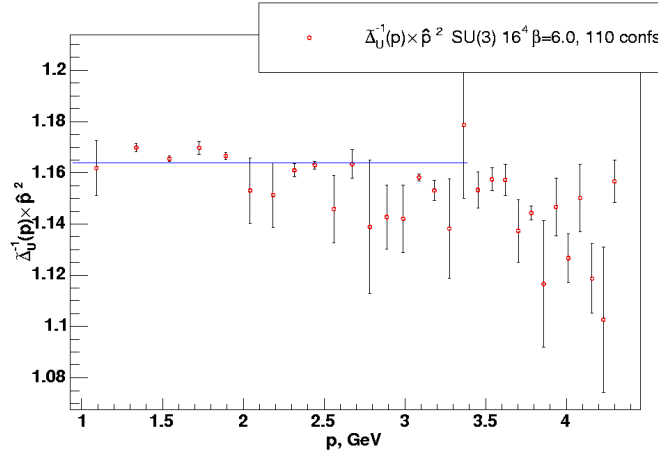


Figure 5: Lattice computation of Δ/Δ_U in Fourier space as a function of the momentum. The statistics is poor for technical reasons. Note also that the region above $\pi/2$ (~ 3 GeV) is affected by strong discretisation effects.

We want to emphasize at this stage that our lattice data both *satisfy* the properly discretised SD equation and *violate* the relation \mathbf{R}_α . The most likely way out we can think of is that the hypothesis $\alpha_\Gamma = 0$ is not verified, i.e. that H_1 is singular when all momenta are small. One more argument in favour of this explanation is provided by a direct numerical examination which shows that, in the eventuality of neglecting the momentum dependence of H_1 , the SD equation is satisfied in the UV but badly violated in the IR. The results of this comparison are given in fig. 6. The details of the method can be found in the appendix.

The possibility of a singular behaviour of H_1 in the infrared has already been considered by various authors, for example in ref. [21] in the framework of exact renormalization group equations. It is known from perturbation theory ([1]), that the sum $H_1 + H_2$ (cf eq. (9)) is exactly equal to 1 at $k = 0$ (this has been checked explicitly in perturbation theory up to fourth order in ref. [19]). This is the argument which is usually called for when advocating $\alpha_\Gamma = 0$. However it does not prevent the contribu-

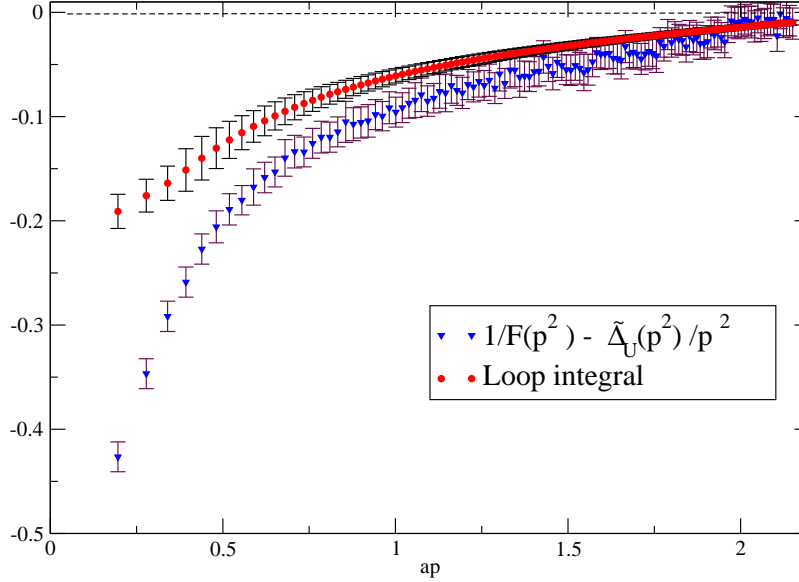


Figure 6: Testing the truncated ghost SD equation on the lattice. The data correspond to a $SU(3)$ simulation at $\beta = 6.4$ and $V = 32^4$, ($a^{-1} = 3.66$ GeV).

tions of the scalar function b to H_1 and H_2 , which cancel out in the sum, from being singular in this limit. Actually, from dimensional considerations, one concludes that b must be of dimension -2 . At $k = 0$ the only dimensional quantity involved (at the perturbative level) is q , which means $b(q, 0; -q) \propto 1/q^2$. This singularity is removed by the kinematical factor in front of b in H_1 and H_2 , but this would no longer be the case for $k \neq 0$ if one had more generally $b(q, k; -q - k) \propto 1/q^2$. *In any case our results appear to plead in favour of a divergent ghost-ghost-gluon vertex in the infrared domain.*

3 Slavnov-Taylor identity and the infrared behaviour of the gluon propagator

Another non-perturbative relation that can be exploited is the Slavnov-Taylor identity. It can be used to constrain (under some hypotheses) the infrared exponent for the gluon dressing function. In the preceding section we have explored the consequences of the very strong assumption that H_1 is regular when all its arguments go to zero and we have shown that this assumption is not tenable when the lattice data are taken into account. We shall now make the weaker hypothesis that the scalar factors present in the decomposition (4) of the ghost-ghost-gluon vertex are regular when one of their arguments go to zero while the others are kept finite and exploit the Slavnov-Taylor identity under this assumption to derive constraints on α_F and α_G .

The Slavnov-Taylor ([1]) identity for the three-gluon function reads

$$\begin{aligned} p^\lambda \Gamma_{\lambda\mu\nu}(p, q, r) &= \frac{F(p^2)}{G(r^2)} (\delta_{\lambda\nu} r^2 - r_\lambda r_\nu) \tilde{\Gamma}_{\lambda\mu}(r, p; q) - \\ &- \frac{F(p^2)}{G(q^2)} (\delta_{\lambda\mu} q^2 - q_\lambda q_\mu) \tilde{\Gamma}_{\lambda\nu}(q, p; r). \end{aligned} \quad (33)$$

One then takes the limit $r \rightarrow 0$ while keeping q and p finite. The tensor structure of $\tilde{\Gamma}_{\mu\nu}(p, k; q)$ has been recalled in eq. (4). We adopt the following notations for particular kinematic configurations:

$$\begin{aligned} a_3(p^2) &= a(-p, p; 0) \\ a_1(p^2) &= a(0, -p; p), \quad b_1(p^2) = b(0, -p; p), \quad d_1(p^2) = d(0, -p; p). \end{aligned} \quad (34)$$

In the present case of one zero momentum the three-gluon vertex may be parameterised in the general way as ([19])

$$\Gamma_{\mu\nu\rho}(p, -p, 0) = (2\delta_{\mu\nu}p_\rho - \delta_{\mu\rho}p_\nu - \delta_{\rho\nu}p_\mu) T_1(p^2) - \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) p_\rho T_2(p^2) + p_\mu p_\nu p_\rho T_3(p^2). \quad (35)$$

Now, exhibiting the dominant part of each term of eq.(33) we obtain:

$$\begin{aligned} T_1(q^2)(q_\mu q_\nu - q^2 \delta_{\mu\nu}) + q^2 T_3(q^2) q_\mu q_\nu + \eta_{1\mu\nu}(q, r) &= \\ \frac{F((q+r)^2)}{G(r^2)} \left[(a_1(q^2) + r_1(q, r)) (\delta_{\mu\nu} r^2 - r_\mu r_\nu) 1 + (b_1(q^2) + r_2(q, r)) q_\mu (r^2 q_\nu - q \cdot r r_\nu) \right. \\ &+ (b_1(q^2) + d_1(q^2) + r_3(q, r)) r_\mu (r^2 q_\nu - q \cdot r r_\nu) \left. \right] + \\ &+ \frac{F((q+r)^2)}{G(q^2)} \left[a_3(q^2) (q_\mu q_\nu - q^2 \delta_{\mu\nu}) + \eta_{2\mu\nu}(q, r) \right] \end{aligned} \quad (36)$$

where $r_{1,2,3}$ and $\eta_{1,2}$ verify

$$\begin{aligned} \lim_{r \rightarrow 0} r_1(q, r) &= \lim_{r \rightarrow 0} r_2(q, r) = \lim_{r \rightarrow 0} r_3(q, r) = 0 \\ \lim_{r \rightarrow 0} \eta_{1\mu\nu}(q, r) &= \lim_{r \rightarrow 0} \eta_{2\mu\nu}(q, r) = 0 \end{aligned} \quad (37)$$

Identifying the leading terms of the scalar factors multiplying the tensors $q_\mu q_\nu$ and $q_\mu q_\nu - q^2 \delta_{\mu\nu}$ we obtain the usual relations ([19]):

$$\begin{aligned} T_1(q^2) &= \frac{F(q^2)}{G(q^2)} a_3(q^2) \\ T_3(q^2) &= 0. \end{aligned} \quad (38)$$

Using these relations we see that Eq.(36) implies:

$$\lim_{r \rightarrow 0} \frac{F(p^2)}{G(r^2)} \left[a_1(q^2) (r^2 \delta_{\mu\nu} - r_\mu r_\nu) + b_1(q^2) (r^2 q_\mu q_\nu - r \cdot q q_\mu r_\nu) \right] = 0 \quad (39)$$

Thus one sees that if $a_1(q^2) \neq 0$ or $b_1 \neq 0$ (and, indeed, one knows from perturbation theory that at large momenta $a_1 = 1$, cf. [1, 19]) (33) can only be compatible with the parameterisation (5) if

$$\alpha_G < 1. \quad (40)$$

We can also, instead of letting $r \rightarrow 0$, take the limit $p \rightarrow 0$ of Eq.(33) as is done in [19]. The dominant part of the l.h.s. of (33) is :

$$(2\delta_{\mu\nu}p \cdot q - p_\mu q_\nu - p_\nu q_\mu)T_1(q^2) - (\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})p \cdot q T_2(q^2) + p \cdot q q_\mu q_\nu T_3(q^2)$$

The r.h.s. is the product of $F(p^2)$ with an expression of at least first order in p . T_1 and T_2 being different from zero we can conclude in this case that $\alpha_F \leq 0$.

Let us repeat here that all these considerations are valid only if all scalar factors of the ghost-ghost-gluon and three-gluons vertices are regular functions when one momentum goes to zero while the others remain finite. Under those hypotheses one obtains important constraints on the gluon and ghost propagators - namely that they are divergent in the zero momentum limit.

3.1 Lattice results

The results for α_G and α_F from lattice simulations are presented in figs. (1) and (7). We present in Table 2 the values of the coefficients for a fit of the form⁷ $(q^2)^\alpha(\lambda + \mu q^2)$.

Group	Volume	β	α_G	α_F
SU(2)	48 ⁴	2.3	1.004 ± 0.015	-0.087 ± 0.015
SU(2)	32 ⁴	2.3	0.968 ± 0.011	-0.109 ± 0.014
SU(3)	32 ⁴	5.75	0.864 ± 0.016	-0.153 ± 0.022

Table 2: Summary of the fitting parameters for the F and G functions

The fits have been performed without using the point $p^2 = 0$ even in the case of the gluon propagator where it is known. Its inclusion would have *forced* α_G to be equal to 1. In any case one sees on fig. 7 that the point at $p^2 = 0$ available in the $SU(2)$, 32⁴ case is compatible with the fit.

For SU(2) and the larger volume the value obtained for α_G is compatible with 1. The situation is less clear for SU(3), but in this case data with the larger volume (48⁴) are lacking. Moreover we have to take into account our experience from previous studies of the gluon propagator where we have always observed that the gluon propagator goes continuously to a finite limit in the infrared region. A very detailed study of the gluon dressing function and specially of its volume dependence at $k = 0$ has been performed by Bonnet et al. (cf ref. [30]). This study shows that a value $\alpha_G = 1$ is compatible with the data (the dressing function shows no signal of discontinuity in the neighbourhood of zero) and that no pathology shows up as the volume goes to infinity.

We conclude that lattice data seem to contradict Zwanziger's result ($G^{(2)}(0) = 0$), [22] and most probably also the predictions derived from the Slavnov-Taylor identity ($G^{(2)}(0)$ infinite). Like in the preceding section a possible way out of this contradiction could consist in dropping the regularity assumptions which have been made in the course of the proof.

⁷A term of the form μq^2 is clearly needed in order to describe a situation like the one in fig. (7 left) where $G^{(2)}(p^2)$ seems to go to a finite limit when p goes to zero

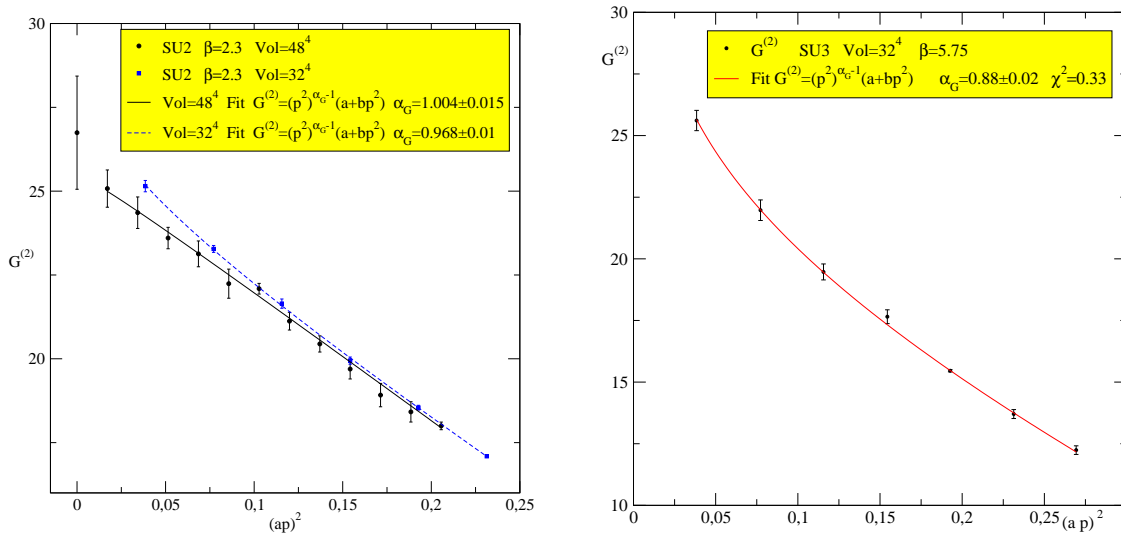


Figure 7: $G^{(2)}(p^2)$ from lattice simulation for $SU(2)$ (left) and $SU(3)$ (right). $\beta_{SU(2)} = 2.3$ and $\beta_{SU(3)} = 5.75$. the volumes are 32^4 and 48^4 for $SU(2)$ and 32^4 for $SU(3)$.

4 Discussion and Conclusions

4.1 Discussion of the validity of SD and ST equations in the infrared

Schwinger-Dyson equations and Slavnov-Taylor identities are valid non-perturbatively. However some care is needed mainly due to Gribov ambiguities in the gauge fixing procedure. One needs a well defined QCD partition function for a given gauge (in the following we concentrate on Landau gauge). Many different non-perturbative prescriptions for the quantisation of non-abelian theories have been suggested : integration only over the absolute minima of the $\int A^2$ functional (Gribov fundamental modular region) [22], summing of copies with signed Faddeev-Popov determinant [9], stochastic quantisation [31], etc. All these prescriptions correspond to different valid gauge fixing procedures.

In lattice numerical simulations two main methods are of practical use: The algorithm which minimises the functional is stopped at its first solution, which is a local minimum, or one takes the smallest local minimum among a given number of trials on the same gauge orbit. One is sure to be inside the Gribov region, never to be inside the fundamental modular region.

The question is whether SD and ST relations are valid in these gauge fixing schemes. It is argued in ([4]) that the Schwinger - Dyson equations are valid under different quantisation prescriptions provided that the Faddeev-Popov determinant vanishes on the boundary of the integration domain. However, the partition functions will differ and hence, the Green functions will be in general different solutions of the the SD equations.

The Slavnov-Taylor identities may be derived from the QCD partition function

using the gauge invariance of the action and stating that the gauge fixed path integral is invariant after a change of variables which corresponds to a gauge transformation. It is then clear that the Slavnov-Taylor identities remain valid whichever gauge fixing procedure has been followed. However this proof has to be taken with care in the presence of singularities of the partition function or of Green functions.

One more general comment is in order. As we have already stated, no Green function with a vanishing ghost momentum can be defined on the lattice since the zero modes of the Faddeev-Popov operator are discarded. For the same reason no source term for the zero mode ghost field is allowed. More generally we do not know of any non perturbative way to define such Green functions. We cannot prove that this means a divergence of the Green functions with one ghost momentum going to zero, but we can suspect that it is the case as lattice indicates for the ghost propagator ⁸. Indeed the close-to-zero modes of the Faddeev-Popov operator are strongly influenced by the Gribov horizon and, for sure, very different from any perturbative result. This casts also some doubt about the use of ST identities or SD equations in the case of a zero ghost momentum.

4.2 Conclusions

We have tried in this work to put together various inputs in order to clarify our understanding of the infrared behaviour of the pure Yang-Mills Green functions. Our findings can be summarised as follows :

1. The lattice Green functions contradict the common *lore* according to which $2\alpha_F + \alpha_G = 0$. The present situation is that $2\alpha_F + \alpha_G > 0$, i.e., the product $F^2(k^2) G(k^2)$ tends to 0 for $k \rightarrow 0$. From what we observe concerning the evolution of the curves with the size of the lattice, it is difficult to imagine how a further increase of the volume could eventually revert this tendency.
2. The result $2\alpha_F + \alpha_G = 0$ (\mathbf{R}_α) which is contradicted by lattice data, is usually claimed to be derived from the ghost SD equation. Our results seem to cast some doubt on the validity of these derivations, based on the assumption of a trivial ghost-ghost-gluon vertex (“naïve approximation”). Indeed, we do verify that the *properly discretised* SD ghost equation on the lattice is well satisfied. We conclude that the lattice data seem to prove that this “naïve approximation” is invalid, and that there exists in the ghost-ghost-gluon vertex a non-perturbative infrared singularity of the form $(k^2)^{\alpha_r}$ which has been neglected in the standard analysis and which leads to the replacement of \mathbf{R}_α by $2\alpha_F + \alpha_G + \alpha_r = 0$ ($\alpha_r < 0$).
3. Regarding the ghost correlator it results from its very definition that it cannot be defined at zero momentum. Its lattice values at low momentum appear to be in favour of a divergent dressing function ($\alpha_F < 0$), as is also suggested by the above theoretical arguments. However the divergence is much slower ($\alpha_F \in [-.15, -.1]$) than what is usually reported. This is in agreement with the conclusions one can draw from the Slavnov Taylor identity.
4. In relation with points 2) and 3) above it is worth insisting on the fact that the Schwinger -Dyson equation by itself is not sufficient to determine the behaviour of

⁸Notice that we cannot claim anything about the behaviour of vertex functions when one ghost momentum goes to zero.

the ghost-propagator and of the ghost-ghost-gluon vertex. Different treatments of the Gribov copies lead to different infrared solutions (see also ref [17] in this respect), *all of which fulfill the SD equation*.

5. As for the gluon propagator the situation is much less clear. Three sources of information are available and give contradictory results.

- The Slavnov Taylor identity, supplemented by regularity assumptions for the ghost-ghost-gluon vertex functions, points towards a *divergent* infrared behaviour.
- Our lattice data indicate a finite limit when the momentum goes to zero. This trend is very clear in the $SU(2)$ case although less compelling for $SU(3)$. The fit has been performed by excluding the point at $p = 0$ but the latter, when known, is compatible with the extrapolated value. These results agree with the ones of our previous studies as well as with the findings of the other lattice groups who have studied this matter ([30]) which include the point $p = 0$ and impose therefore $\alpha_G = 1$.
- Zwanziger's result [22] states that the gluon propagator vanishes at $k=0$ but a fully satisfactory proof of a *continuous* vanishing as $k \rightarrow 0$ is still lacking.

We are unable for the moment to settle this point in a totally unambiguous way.

This set of conclusions raises some questions. First of all further studies are still under way in order to fix the issues related to point 4). As to point 2), a direct lattice study of the ghost-ghost-gluon vertex at low momenta is desirable, although difficult. Such a study has recently been performed for $SU(2)$ in a specific kinematical situation (zero-momentum gluon) ([32]). The precise relevance of this special case to the points we have considered remains to be clarified.

Note added

After the completion of this work we realised that the particular situation ($\alpha_F = 0$) which we have mentioned in the fourth column of table 1 but not fully investigated might provide a good agreement with the lattice data while complying with the constraints stemming from the SD equation. This possibility is discussed in a further publication ([35]). Let us stress that this solution too is compatible with the non-vanishing of $2\alpha_F + \alpha_G$.

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6 Appendix: Testing the naive approximation of the ghost SD equation

The simplest approximation of the ghost SD equation (10) corresponds to the case $H_1(q, k) = 1 \forall q, k$:

$$\frac{1}{F(k)} = 1 + \frac{g_0^2 N_c}{k^2} \int \frac{d^4 q}{(2\pi)^4} \left(\frac{F(q^2) G((k-q)^2) (k \cdot q)^2 - k^2 q^2}{q^2 (k-q)^2 (q-k)^2} \right) \quad (41)$$

Strictly speaking this equation, written in this way, is meaningless since it involves UV divergent quantities but a corresponding meaningful renormalized version can be given (see the caveats about eq (7) in subsection (2.1)). We want to check whether lattice propagators satisfy it. According to perturbation theory, it should be true at large k . Lattice propagators are discrete functions, and thus one has to handle the problem of the numerical evaluation of the loop integral I in (41). Let us express the integrand in terms of q^2 and $(k-q)^2$

$$I = \int \frac{d^4 q}{(2\pi)^4} \frac{F^2(q) G(k-q)}{q^2 (k-q)^2} \left[\frac{(k-q)^2}{4} + \frac{(k^2)^2 + (q^2)^2 - 2k^2 q^2}{4(k-q)^2} - \frac{q^2 + k^2}{2} \right]. \quad (42)$$

Then we write

$$I = I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

each I_i corresponds to one term in (42).

All these integrals have the form

$$I_i = C_i(k) \int \frac{d^4 q}{(2\pi)^4} f_i(q) h_i(k-q).$$

The convolution in the r.h.s. is just the Fourier transform of the product at the same point in configuration space:

$$\int \frac{d^4 q}{(2\pi)^4} f_i(q) h_i(k-q) = F_+ \left(F_-(f_i)[x] F_-(h_i)[x] \right) (k),$$

where $F_-(\hat{f})(x)$ is an inverse and $F_+(f)(k)$ a direct Fourier transform. Thus, in order to calculate the integral I from discrete lattice propagators one proceeds as follows:

1. calculate $\{f_i\}(p)$ and $\{h_i\}(p)$ as functions of $F(p), G(p), p^2$ for all i
2. apply the inverse Fourier transform F_- to all these functions and get $f_i(x)$ and $h_i(x)$
3. compute the product at the same point $f_i(x) \cdot h_i(x)$
4. apply the direct Fourier transform F_+ to $f_i(x) \cdot h_i(x)$

The calculation of Fourier transforms involves a Hankel transformation which is numerically evaluated by means of a Riemann sum

$$f(r) = (2\pi)^{-2} \|r\|^{-2} \sum_{i=1}^N J_1(r\rho_i) \rho_i^2 \frac{\hat{f}[\rho_i] + \hat{f}[\rho_{i-1}]}{2} (\rho_i - \rho_{i-1}), \quad \rho_0 = 0. \quad (43)$$

The inverse transformation is done in the similar way. In practice, because of the lattice artefacts which become important at large ρ the summation has to be restricted to $\rho < \rho_{max} \simeq 2.2$ instead of the “ideal” value 2π .

Errors There are three important sources of errors : statistical Monte-Carlo errors for $F(q^2)$ and $G(q^2)$, the bias due to the integral discretization and the truncation of the ρ -summation to values lower than ρ_{max} . The second one is dominated by the neglected contribution coming from the UV cut-off of the integral (the integration is performed on some ball $B(0, L)$ instead of \mathbb{R}^4). Let us estimate the error on the Fourier transform of the product in such a case:

$$F_+(f(x)g(x))(k) = \int_{B(0,L)} d^4p d^4q \delta_\epsilon(k-p-q) \hat{f}(p) \hat{g}(q), \quad (44)$$

where the ϵ -approximation to delta function is $\delta_\epsilon(p) = \int_{B(0,L)} d^4x e^{i(x,p)}$. Considering ϵ small enough we can integrate on q around the point $(k-p)$, obtaining finally:

$$(f \star g)_L \approx (f \star g)_\infty + \text{Vol}(B(0, \epsilon(L))) \cdot (f \star \nabla g)_\infty$$

This gives us an estimation of the error coming from the UV cut-off. As for the last source of error, it may be neglected because of the following argument : the integral is logarithmically divergent, therefore the neglected behaves as $\log\left(\frac{2\pi}{La}\right) - \log\left(\frac{\rho_{max}}{La}\right) = \log\left(\frac{2\pi}{\rho_{max}}\right)$. Thus it remains finite as a goes to zero and gets smaller and smaller as compared to the part actually computed.

Results: We still have to face the same problem we have already encountered in section (2.4.3), namely that the lattice Faddeev-Popov operator involves the non trivial discretisation Δ_U of the Laplacian operator. This is taken into account by means of the substitution of $\widetilde{\Delta_U}(p^2)/p^2$ to the “1” term in the l.h.s of equation (41) We present on (Fig. 6) the result of the numerical integration described above. We have chosen for this purpose the data set from the simulation with the gauge group $SU(3)$ at $\beta = 6.4, V = 32^4$. One sees that the equality is achieved at large momenta, but in the infrared the naive approximation of the ghost SD equation fails.

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